# Lecture 12

## 11.3. Coupled Oscillatory Circuits

### 11.3.1. Some General Concepts of Coupled Circuits

The frequency selective properties of single oscillatory circuits are not high ($K_B = 0{,}1$). A higher selectivity can be reached by the use of a system of coupled circuits. Oscillatory circuits are called coupled if the processes in them influence one another.

There are systems of two and more coupled circuits. A circuit to which a signal source is connected is called a primary tuned circuit; a circuit from which a signal is output is called a secondary tuned circuit.

A primary tuned circuit is connected with a secondary one by a coupling element (Fig. 11.24).

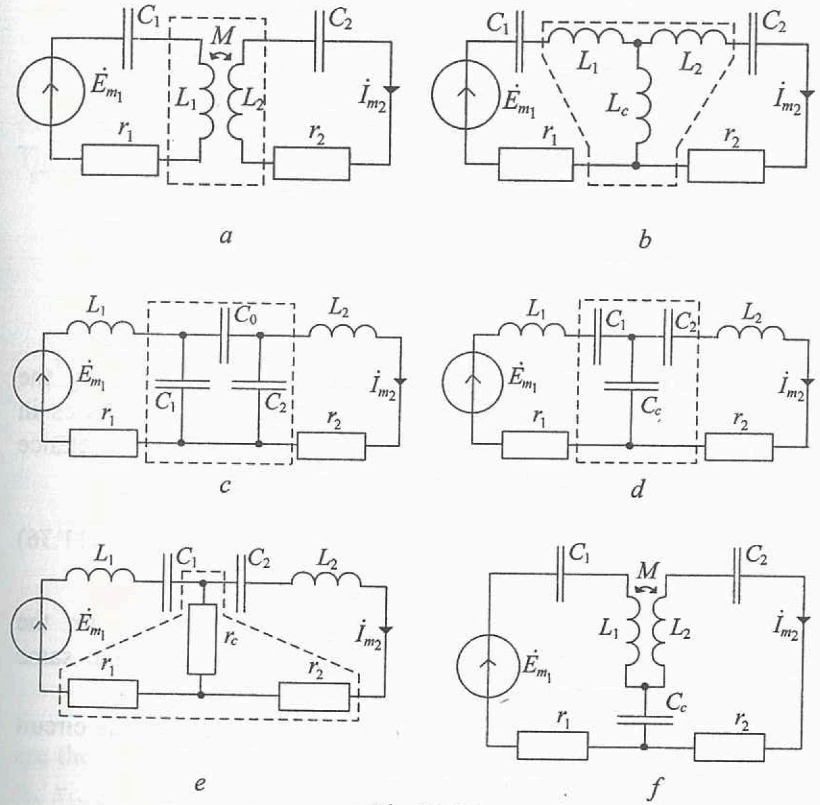


Fig. 11.24

Here are circuits with inductive coupling (Fig. 11.24, *a*), with conductive coupling (Fig. 11.24, *b*), with external capacitive coupling (Fig. 11.24, *c*), with internal capacitive coupling (Fig. 11.24, *d*), with resistive coupling (Fig. 11.24, *e*), and with combined coupling (Fig. 11.24, *f*).

A system of coupled circuits with any type of coupling can be represented as an equivalent T-circuit (Fig. 11.25, *a*) or П-circuit. In a T-circuit, $Z_1'$ and $Z_2'$ are the complex impedances of the primary and secondary circuits without taking into account the coupling impedance $Z_c$. In a П-circuit, $Y_1'$ and $Y_2'$ are the complex conductances of the primary and secondary nodes 1 and 2 without taking into account the coupling conductance $Y_0$. $Z_c$ is an impedance that is common for both circuits. $Y_0$ is a conductance that is common for both nodes. For sufficiently loose coupling, we may consider that $Z_c \approx \dfrac{1}{Y_0}$.

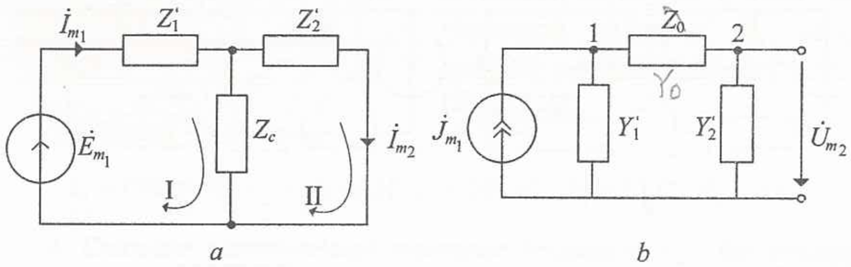



Fig. 11.25

For quantitative estimation of the degree of circuit coupling the coupling coefficient is used. Most often, in order to reduce losses in transferring energy from the primary to the secondary circuit, reactance is used as a coupling element. Then the coupling coefficient is:

$$K = \frac{x_c}{\sqrt{x_1 x_2}}, \qquad (11.36)$$

where $x_c$ is the reactance of the coupling element; $x_1$, $x_2$ are the reactances of the primary and secondary circuits being of the same nature as the coupling element.

It is clear that $0 \le K \le 1$. At resonance, the reactances of the circuit elements are equal to the characteristic impedances:

$$x_1 = x_{L1} = x_{c1} = \rho_1; \qquad x_2 = x_{L2} = x_{c2} = \rho_2.$$



Then, according to (11.36)

$$K = \frac{x_c}{\sqrt{\rho_1 \rho_2}}.$$

According to the degree of coupling there are the following circuits:

- very loosely coupled ($K \le 0{,}01$);
- loosely coupled ($0{,}01 < K \le 0{,}05$);
- tightly coupled ($0{,}05 < K \le 0{,}9$);
- very tightly coupled ($0{,}9 < K \le 1$).

Most often systems of coupled circuits operate as loosely coupled. Reactance serves as a coupling element. Resistive coupling is used when frequency-independent coupling between circuits is necessary.

### *11.3.2. Equivalent Circuits of a System of Two Coupled Oscillatory Circuits*

Consider a simplest system of two coupled circuits using as an example the equivalent T-circuit shown in Fig. 11.25, *a*.
The system of loop equations for the loops I and II can be written as:

$$\begin{vmatrix} Z_1' + Z_c & -Z_c \\ -Z_c & Z_2' + Z_c \end{vmatrix} \begin{vmatrix} \dot{I}_{m1} \\ \dot{I}_{m2} \end{vmatrix} = \begin{vmatrix} \dot{E}_{m1} \\ 0 \end{vmatrix}. \qquad (11.37)$$

Let us introduce the designations: $Z_1 = Z_1' + Z_c$, $Z_2 = Z_2' + Z_c$.

From (11.37):
$$\begin{vmatrix} Z_1 & -Z_c \\ -Z_c & Z_2 \end{vmatrix} \begin{vmatrix} \dot{I}_{m1} \\ \dot{I}_{m2} \end{vmatrix} = \begin{vmatrix} \dot{E}_{m1} \\ 0 \end{vmatrix}.$$

Here

$$Z_1 = r_1 + jx_1 = r_1\left(1 + j\frac{x_1}{r_1}\right) = r_1(1 + j\xi_1);$$

$$Z_2 = r_2 + jx_2 = r_2\left(1 + j\frac{x_2}{r_2}\right) = r_2(1 + j\xi_2)$$

are the complex impedances of the primary and secondary circuits; $r_1$, $r_2$, $x_1$, $x_2$, $\xi_1$, $\xi_2$ are the resistances, reactances, and the generalized detunings of the primary and secondary circuits.

Solving (11.37) we get:

$$\dot{I}_{m1}=\frac{\Delta_1}{\Delta}=\frac{Z_2\dot{E}_{m1}}{Z_1Z_2-Z_c^2}=\frac{\dot{E}_{m1}}{Z_1-\frac{Z_c^2}{Z_2}}=\frac{\dot{E}_{m1}}{Z_1+Z_{1in}}=\frac{\dot{E}_{m1}}{Z_{1e}};\qquad(11.38)$$

$$\dot{I}_{m2}=\frac{\Delta_2}{\Delta}=\frac{Z_c\dot{E}_{m1}}{Z_1Z_2-Z_c^2}=\frac{\frac{Z_c}{Z_1}\dot{E}_{m1}}{Z_2-\frac{Z_c^2}{Z_1}}=\frac{\dot{E}_{m2}}{Z_2+Z_{2in}}=\frac{\dot{E}_{m2}}{Z_{2e}}.\qquad(11.39)$$

Here

$$Z_{1e}=Z_1+Z_{1in}=r_{1e}+jx_{1e},$$
$$Z_{2e}=Z_2+Z_{2in}=r_{2e}+jx_{2e}$$

are the equivalent impedances of the primary and secondary circuits;

$$Z_{1in}=-\frac{Z_c^2}{Z_2}=r_{1in}+jx_{1in};\quad Z_{2in}=-\frac{Z_{in}^2}{Z_1}=r_{2in}+jx_{2in}\qquad(11.40)$$

are the impedances inserted into the primary (from the side of the secondary) and secondary (on the side of the primary) circuits.

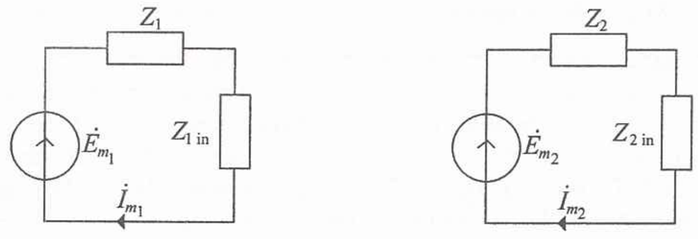
$\dot{E}_{m2}=\frac{Z_c}{Z_1}\dot{E}_{m1}$ is the EMF induced in the secondary circuit from the side of the primary circuit.

From (11.38) and (11.39) we have:

$$\dot{I}_{m2}=\frac{\Delta_2}{\Delta_1}\dot{I}_{m1}=\frac{Z_c\dot{E}_{m1}}{Z_2\dot{E}_{m1}}\dot{I}_{m1}=\frac{Z_c}{Z_2}\dot{I}_{m1}.$$

Using (11.38) and (11.39) we can build a single-loop equivalent circuit of a system of two coupled circuit. So, according to (11.38) we get a circuit based on the current in the primary circuit (Fig. 11.26, *a*); (11.39) and a circuit based on the current in the secondary circuit (Fig. 11.26, *b*).






*a* *b*

Fig. 11.26

That is the system consisting of two coupled circuits is replaced by one circuit. The impedance of the other circuit is taken into account by inserting impedances. That is why this method of calculation is called the method of inserted impedances. Let us consider the inserted impedances (11.40). Since most often $r_c = 0$, $Z_c = jx_c$, then, from (11.40):

$$Z_{1in}=-\frac{Z_c^2}{Z_2}=\frac{x_c^2}{r_2+jx_2}=r_2\left(\frac{x_c}{z_2}\right)^2-jx_2\left(\frac{x_c}{z_2}\right)^2=r_{1in}+jx_{1in};$$

$$Z_{2in}=\frac{Z_c^2}{Z_1}=\frac{x_c^2}{r_1+jx_1}=r_1\left(\frac{x_c}{z_1}\right)^2-jx_1\left(\frac{x_c}{z_1}\right)^2=r_{2in}+jx_{2in}.$$

Here $r_{1in}=r_2\left(\frac{x_c}{z_1}\right)^2$, $r_{2in}=r_1\left(\frac{x_c}{z_1}\right)^2$ — active components of the inserted impedances. They take into account losses introduced into the circuit from the other circuit.

$$x_{1in}=-x_2\left(\frac{x_c}{Z_2}\right)^2;\quad x_{2in}=-x_1\left(\frac{x_c}{Z_1}\right)^2;\qquad(11.41)$$

$$Z_{1e}=r_{1â}+jx_{1â}=Z_1+Z_{1in}=r_1+jx_1+r_{1in}+jx_{1in}=$$
$$=r_1+r_{1in}+j(x_1+x_{1in})=r_1+r_2\left(\frac{x_c}{Z_2}\right)^2+j\left[x_1-x_2\left(\frac{x_c}{Z_2}\right)^2\right].\qquad(11.42)$$

### *11.3.3. Resonance in a System of Coupled Circuits*

Resonance in a system of coupled oscillatory circuits is a phenomenon wherein the amplitudes of currents in the circuits or voltages across the elements reach their maximum values. If there are no losses in the circuits the oscillation amplitudes tend to infinity. For a generalized T-circuit (Fig. 11.25, *a*) it corresponds to the system's determinant Δ in (11.38) and (11.39) being equal to zero:

$$\Delta = Z_1 Z_2 - Z_c^2 = 0.$$

If there are losses, the currents and voltages of different circuits reach their maximum values at different frequencies. Therefore, we cannot speak have about one resonance frequency for the entire system of coupled circuits.

As far as high-*Q* circuits are concerned, damping can be neglected in them. In this case it is said that there is a single resonance frequency for the entire system of coupled circuits. It is this kind of circuits that we will consider below.

There are two kinds of resonance depending on the components of which circuit the tuning to resonance is done for a specified frequency of the input signal:

– particular resonances (the first, second, and the main or individual);

– complex resonances (the first, second, and total).

Most often the signal output for further use is the current in the second circuit. Therefore, tuning a system of two coupled circuits to resonance refers to selection of the set of circuit parameters that yields the maximum current value in the second circuit.

The first particular resonance is achieved by varying the parameters of the reactive components of the primary circuit only (usually *C*1):

$$x_1 = \text{var}, \quad x_2 = \text{const}, \quad x_c = \text{const}.$$

The resonance is achieved by satisfying the condition

$$x_{1e} = x_1 + x_{1in} = x_1 - x_2\left(\frac{x_{in}}{z_2}\right)^2 = 0.$$

That is

$$x_1 = -x_{1in} = x_2\left(\frac{x_{in}}{z_2}\right)^2. \qquad (11.43)$$



Determine the current in the secondary circuit in the case of the first particular resonance. Taking account of (11.38) we get:

$$\dot{I}_{m2M1} = \frac{Z_c}{Z_2}\dot{I}_{m1M1} = \frac{Z_c \dot{E}_{m1}}{Z_2 Z_{1e}}.$$

Taking into account that $r_{cb} = 0$ and keeping in mind (11.42):

$$\dot{I}_{m2M1} = \frac{Z_c}{Z_2}\dot{I}_{m1M1} = \frac{x_c \dot{E}_{m1}}{z_2\sqrt{r_{1e}^2 + x_{1e}^2}}.$$

At resonance, according to (11.43) and taking into account (11.44):

$$\dot{I}_{m2M1} = \frac{x_c \dot{E}_{m1}}{Z_2 r_{1e}} = \frac{x_c \dot{E}_{m1}}{Z_2(r_1 + r_{1in})} = \frac{x_c \dot{E}_{m1}}{Z_2\left[r_1 + r_2\left(\frac{x_c}{Z_2}\right)^2\right]}. \qquad (11.44)$$

An example of first particular resonance application is tuning the output oscillatory circuit of a UHF oscillator the secondary circuit of which, being inductively coupled with the primary circuit, is the transmitting aerial *A* ($r_a$, $C_a$) (Fig. 11.27).

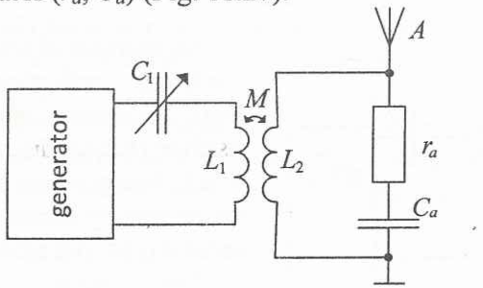



Fig. 11.27

*The second particular resonance* is achieved by varying the parameters of the reactive components of the secondary circuit only (usually $C_2$):

$$x_1 = \text{const}, \quad x_2 = \text{var}, \quad x_c = \text{const}.$$

The resonance is achieved by satisfying the condition.

$$x_{2e} = x_2 + x_{2in} = x_2 - x_1\left(\frac{x_c}{Z_1}\right)^2 = 0.$$

That is

$$x_2 = -x_{2in} = x_1\left(\frac{x_c}{z_1}\right)^2. \qquad (11.45)$$

Determine the current in the secondary circuit in the case of the second particular resonance. From (11.39) we get

$$\dot{I}_{m2M2} = \frac{\dot{E}_{m2}}{Z_{2e}} = \frac{Z_c \dot{E}_{m1}}{Z_1 Z_{2e}}.$$

For $r_c = 0$:

$$\dot{I}_{m2M2} = \frac{x_c \dot{E}_{m1}}{z_1\sqrt{r_{2e}^2 + x_{2e}^2}}.$$

At resonance, according to (11.45)

$$\dot{I}_{m2M2} = \frac{x_c \dot{E}_{m1}}{z_1 r_{2e}} = \frac{x_c \dot{E}_{m1}}{z_1(r_2 + r_{2in})} = \frac{x_c \dot{E}_{m1}}{Z_1\left[r_2 + r_1\left(\frac{x_c}{Z_1}\right)^2\right]}. \qquad (11.46)$$

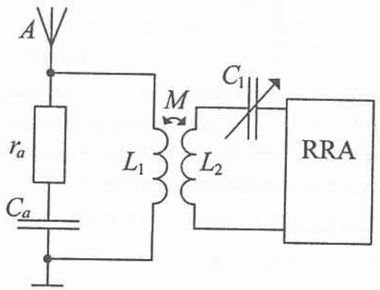



Fig. 11.28

An example of secondary particular resonance application is tuning the input oscillatory circuit of a radio receiver (RRA), the primary circuit of which, being inductively coupled with the secondary circuit, is the receiving aerial $A$ ($r_a$, $C_a$) (Fig. 11.28).

*The main or individual resonance* is achieved by varying the parameters of both circuits (usually $C_1$ and $C_2$), i.e.

$$x_1 = \text{var}, \quad x_2 = \text{var}, \quad x_c = \text{const} \approx 0.$$

In other words, both circuits are taken separately and tuned to resonance individually.

The resonance is achieved under the condition:

$$x_{1e} = 0, \quad x_{2e} = 0.$$



From (11.43) and (11.45) we can see that this condition is also satisfied when

$$x_1 = -x_{1in} = x_2 = -x_{2in} = 0. \qquad (11.47)$$

From (11.44), taking account of (11.47), we get:

$$\dot{I}_{m2M} = \frac{x_c \dot{E}_{m1}}{r_2 r_{1e}} = \frac{x_c \dot{E}_{m1}}{r_2\left[r_1 + r_2\left(\frac{x_c}{r_2}\right)^2\right]} = \frac{x_c \dot{E}_{m1}}{r_1 r_2 + x_c^2}. \qquad (11.48)$$

Similarly, from (11.46) we obtain

$$\dot{I}_{m2M} = \frac{x_c \dot{E}_{m1}}{r_1 r_{2e}} = \frac{x_c \dot{E}_{m1}}{r_1\left[r_2 + r_1\left(\frac{x_c}{r_1}\right)^2\right]} = \frac{x_c \dot{E}_{m1}}{r_1 r_2 + x_c^2}. \qquad (11.49)$$

Comparing the results of (11.48) with (11.44) and the results of (11.49) with (11.46) we can see that, with the same numerators, the denominators in (11.48) and (11.49) are considerably smaller than in (11.44) and (11.46) because $r_1 \ll z_1$, $r_2 \ll z_2$. That is, in the case of the main resonance, a considerably larger current is achieved in the secondary circuit than in the cases of the first and second particular resonances. Therefore, if the design of a real device permits, the circuits are usually tuned individually so that the condition of the main resonance may be provided in the system of coupled circuits.

Consider the complex resonances.

From expressions (11.44), (11.46), (11.48), (11.49) for the secondary circuit current at particular resonances we can see that the current $\dot{I}_{m2}$ can be increased by selecting the impedance $x_c$.

*The first complex resonance* is achieved by satisfying the conditions of the first particular resonance and selecting the optimal coupling impedance, i.e.

$$x_{1e} = 0, \qquad x_c = x_{copt1}.$$

Determine $x_{copt1}$. Analyze the function $I_{m2M1} = f(x_c)$ in (11.44) for extremes.

$$\frac{\partial I_{m2\mathrm{M1}}}{\partial x_c}=\frac{\dot{E}_{m1}}{z_2}\frac{r_1+r_2\left(\dfrac{x_c}{z_2}\right)^2-2r_2\left(\dfrac{x_c}{z_2}\right)^2}{\left[r_1+r_2\left(\dfrac{x_c}{z_2}\right)^2\right]^2}=0.$$

Hence

$$r_1+r_2\left(\frac{x_c}{z_2}\right)^2-2r_2\left(\frac{x_c}{z_2}\right)^2=0.$$

Now, the optimal coupling impedance is:

$$r_{\mathrm{copt1}}=Z_2\sqrt{\frac{r_1}{r_2}}. \qquad (11.50)$$

Substituting (11.50) to (11.44) we get the maximum current in the secondary circuit

$$\dot{I}_{m2MM}=\frac{Z_2\sqrt{\dfrac{r_1}{r_2}}\,\dot{E}_{m1}}{Z_2\left(r_1+r_2\dfrac{Z_2^2 r_1}{r_2 Z_2^2}\right)}=\frac{\dot{E}_{m1}}{2\sqrt{r_1+r_2}}. \qquad (11.51)$$

*The second complex resonance* is achieved by satisfying the conditions of the second particular resonance and selecting the optimal coupling impedance, i.e.

$$x_{2e}=0, \qquad x_c=x_{\mathrm{copt2}}.$$

Determine $x_{\mathrm{copt2}}$. Analyze the function $I_{m2\mathrm{M2}}=f\,(x_c)$ in (11.46) for extremes:

$$\frac{\partial I_{m2\mathrm{M2}}}{\partial x_c}=\frac{\dot{E}_{m1}}{z_1}\frac{r_2+r_1\left(\dfrac{x_c}{z_1}\right)^2-2r_1\left(\dfrac{x_c}{z_1}\right)^2}{\left[r_2+r_1\left(\dfrac{x_c}{z_1}\right)^2\right]^2}=0.$$

From here

$$r_2+r_1\left(\frac{x_c}{z_1}\right)^2-2r_1\left(\frac{x_c}{z_1}\right)^2=0.$$



Now, the optimal coupling impedance is:

$$r_{\mathrm{copt2}}=Z_1\sqrt{\frac{r_2}{r_1}}. \qquad (11.52)$$

Substituting (11.52) into (11.46) we get the maximum current in the secondary circuit.

$$\dot{I}_{m2M2}=\frac{Z_1\sqrt{\dfrac{r_1}{r_2}}\,\dot{E}_{m1}}{Z_1\left(r_2+r_1\dfrac{Z_1^2 r_2}{r_1 Z_1^2}\right)}=\frac{\dot{E}_{m1}}{2\sqrt{r_1 r_2}}. \qquad (11.53)$$

*The total resonance* is achieved by satisfying the conditions of the main resonance and selecting the optimal coupling impedance, i.e.

$$x_{1e}=0,\quad x_{2e}=0,\quad x_c=x_{\mathrm{copt}}.$$

Determine $x_{\mathrm{copt}}$. Analyze the function $I_{m2\mathrm{M2}}=f\,(x_c)$ in (11.48) for extremes:

$$\frac{dI_{m2\mathrm{M}}}{dx_c}=\dot{E}_{m1}\frac{r_1r_2+x_c^2-2x_c^2}{\left(r_1r_2+x_c^2\right)^2}=0.$$

Hence

$$r_1r_2-x_c^2=0.$$

Now, the optimal coupling impedance is:

$$r_{\mathrm{copt}}=\sqrt{r_1r_2}. \qquad (11.54)$$

Substituting (11.54) into (11.49) we get the maximum current in the secondary circuit:

$$\dot{I}_{m2MM}=\frac{\sqrt{r_1r_2}\,\dot{E}_{m1}}{r_1r_2+r_1r_2}=\frac{\dot{E}_{m1}}{2\sqrt{r_1r_2}}. \qquad (11.55)$$

Thus, at all complex resonances, the same value of current in (11.51), (11.53), (11.55) is achieved in the secondary circuit. However, when tuning to the first and second complex resonances, changing the coupling impedance contradicts the conditions of the corresponding particular resonances and causes detuning of the circuits on these resonances. Therefore, after tuning to a complex resonance by means of $x_c$ it is necessary to adjust the circuit to the corresponding particular resonance by means of $C_1$ or $C_2$. This complicates the tuning as it is performed by the method of successive approximations. In tuning to a total resonance, a step-by-step tuning is not needed since tuning to the main resonance of the primary and secondary circuits is done individually at $x_c = 0$, i.e. it does not depend on $x_c$.

Compare the optimal coupling impedances (11.50), (11.52) and (11.54):

$$\frac{r_{copt1}}{r_{copt}} = \frac{z_2\sqrt{\frac{r_1}{r_2}}}{\sqrt{r_1 r_2}} = \frac{z_2}{r_2} > 1, \text{ thus } r_{copt1} > r_{copt};$$

$$\frac{r_{copt2}}{r_{copt}} = \frac{z_1\sqrt{\frac{r_2}{r_1}}}{\sqrt{r_1 r_2}} = \frac{z_2}{r_1} > 1, \text{ thus } r_{copt2} > r_{copt}.$$

Thus, at a total resonance the least coupling is needed (Fig. 11.29).

Consider the power in the circuits. When a system of coupled circuits is represented by its equivalent circuit according to Fig. 11.26, *b*, the resistance of the secondary circuit is

$$r_2 = r_{k2} + r_{l\,int},$$

where $r_{k2}$ is the internal resistance of the circuit; $r_{l\,int}$ is the introduced load impedance.

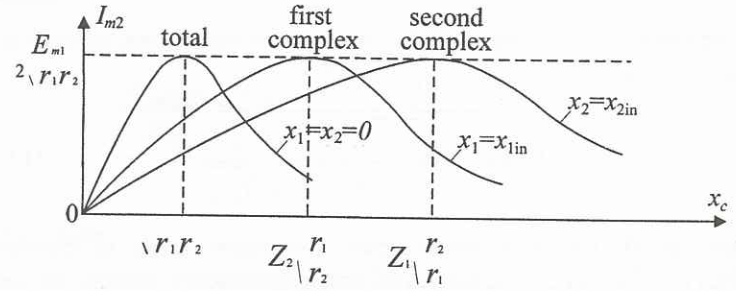



Fig. 11.29



Usually $r_{k2} << r_{l\,int}$, that is $r_2 \approx r_{l\,int}$. Then, at complex resonances an active power develops in the load.

$$P_{lm} = I^2_{2mm} r_l = I^2_{2mm} r_2 = \left(\frac{E_1}{2\sqrt{r_1 r_2}}\right)^2 r_2 = \frac{E_1^2}{4r_1}.$$

The resistance $r_{2in}$ in Fig. 11.21, *b* can be obtained from (11.41) taking onto account (11.54) and (11.47) at a total resonance, i.e.

$$r_{2in} = r_1\left(\frac{r_c}{z_1}\right)^2 = r_1\left(\frac{\sqrt{r_1 r_2}}{r_1}\right)^2 = r_2.$$

That is, the condition of maximum active power transmission from the source to the load is observed

$$r_{2in} = r_{l\,int}.$$

That is why the power in (11.56) is equal to the total power supplied by the source and the efficiency is equal to 50 %.

The optimal coupling impedance obtained earlier in (11.54) cannot be obtained in real circuits. The closeness of a real coupling to the optimal one is evaluated by the coupling factor:

$$A = \frac{x_c}{x_{copt}} = \frac{x_c}{\sqrt{r_1 \cdot r_2}}. \qquad (11.56)$$

Taking into account the coupling coefficient between the circuits we get:

$$x_c = K\sqrt{\rho_1\rho_2}.$$

Then

$$A = \frac{K\sqrt{\rho_1\rho_2}}{\sqrt{r_1 r_2}} = K\sqrt{Q_1 Q_2}.$$

The coupling coefficient is:

$$K = \frac{A}{\sqrt{Q_1 Q_2}} = A\sqrt{d_1 d_2}.$$

For the optimal coupling ($A = 1$)

$$K_{opt} = \frac{1}{\sqrt{Q_1 Q_2}} = \sqrt{d_1 d_2}.$$

For ideal circuits ($Q_1 = Q_2 = Q$)

$$K_{\text{opt}} = \frac{1}{Q} = d.$$

We can see that the higher the $Q$-factor of a tuned circuit, the lower the optimal coupling coefficient should be. For $Q$ = 100,200 it is a fraction of 1 per cent.

### 11.3.4. Frequency Characteristics of a System of Coupled Circuits

Consider the complex transmission conductance as a circuit complex function.

$$Y_{21}(j\omega) = \frac{I_{m2}(j\omega)}{E_{m1}(j\omega)} = \frac{\dot{I}_{m2}}{\dot{E}_{m1}}.$$

From (11.39) we get

$$Y_{21}(j\omega) = \frac{Z_{back}}{Z_1 Z_2 - Z_{back}^2}.$$

With $Z_{back} = j \cdot x_{back}$ we get

$$Y_{21}(j\omega) = \frac{jx_c}{r_1(1+j\xi_1)r_2(1+j\xi_2)+x_c^2} =$$

$$= \frac{j\dfrac{2x_c}{\sqrt{r_1 r_2}} \cdot \dfrac{1}{2\sqrt{r_1 r_2}}}{(1+j\xi_1)(1+j\xi_2)+\dfrac{x_c}{r_1 r_2}}. \tag{11.57}$$

Since from (11.51)

$$\frac{1}{2\sqrt{r_1 r_2}} = \frac{\dot{I}_{m2MM}}{\dot{E}_{m1}} = Y_{21mm},$$

then, taking account of (11.56), from (11.57) we get

$$Y_{21}(j\omega) = \frac{j\dot{I}_{m2MM}\, 2A}{\dot{E}_{m1}\left[(1+j\xi_1)(1+j\xi_2)+A^2\right]}.$$



The normalized transfer admittance is:

$$Y_{21n}(j\omega) = \frac{Y_{21}(j\omega)}{Y_{21MM}(j\omega)} = \frac{2A}{1+A^2-\xi_1\xi_2+j(\xi_1+\xi_2)}. \tag{11.58}$$

According to (11.58), the normalized admittance or normalized current depends on the detunings $\xi_1$, $\xi_2$. Therefore, the amplitude-frequency characteristic is a set of surfaces $Y_{21n}(\omega) = f(\xi_1, \xi_2)$. The plane with the axes $\xi_1$, $\xi_2$ is called the detuning plane.

Cuttings of the surfaces $Y_{21n}(\omega) = f(\xi_1, \xi_2)$ by planes perpendicular to the detuning plane give the amplitude-frequency characteristics (AFC).

In the case of non-identical circuits ($\xi_1 \neq \xi_2$), these curves are asymmetrical, which corresponds to distortion of signals as they pass through the circuit. That's why using identical circuits ($\xi_1 = \xi_2 = \xi$) is preferable.

Then

$$Y_{21n}(j\omega) = Y_{21n}(\omega)e^{j\varphi(\omega)} = \frac{2A}{1+A^2-\xi^2+2j\xi},$$

where

$$Y_{21n}(\omega) = \frac{2A}{\sqrt{\left(1+A^2-\xi^2\right)^2+4\xi^2}} \tag{11.59}$$

— amplitude-frequency characteristic; $\varphi(\omega) = -\text{atan}\dfrac{2\xi}{1+A^2-\xi^2}$ — phase-frequency characteristic.

In order to determine the resonance frequencies let us study the function $Y_{21n}(\omega) = f(\xi_1, \xi_2)$ for extremes:

$$\frac{dY_{21n}(\xi)}{d\xi} = 2A\left[\frac{1}{\left(1+A^2-\xi^2\right)^2+4\xi^2} \cdot \frac{1}{2\sqrt{\left(1+A^2-\xi^2\right)^2+4\xi^2}}\right].$$

That is

$$2\left[\left(1+A^2-\xi^2\right)(-2\xi)+8\xi\right]=0,$$

$$\xi\left(1-A^2+\xi^2\right)=0. \tag{11.60}$$

Solving (11.60) we get

$$\xi_0 = 0; \quad \xi_{12} = \pm\sqrt{A^2 - 1}\,. \tag{11.61}$$

So, from (11.12) and (11.13):

$$\omega_{res} \approx \omega_0 \left(1 + \frac{1}{2Q}\xi\right), \tag{11.62}$$

then at $\xi_0 = 0$ we get one resonance frequency

$$\omega_{res0} = \omega_0 = \frac{1}{\sqrt{LC}} \tag{11.63}$$

at $\xi_{1,2} = \pm\sqrt{A^2 - 1}$ we get another two resonance frequencies

$$\omega_{res1,2} = \omega_0 \left(1 \pm \frac{1}{2Q}\sqrt{A^2 - 1}\right). \tag{11.64}$$

The analysis of the expressions obtained shows that the number of extremes in the AFC depends on the coupling factor $A$. Let us consider possible states.

1. $A < 1$. The coupling is looser than optimal. From (11.61) we get one solution $\xi_0 = 0$. That is, according to (11.62) and (11.63), a resonance is achieved at one frequency $\omega_0 = \frac{1}{\sqrt{LC}}$. From (11.59), at $\xi = 0$ we get:

$$Y_{21n}(\omega_0) = \frac{2A}{1 + A^2}\,. \tag{11.65}$$

For A < 1 we get $Y_{21n}(\omega_0) < 1$, the current in the secondary circuit does not reach a maximum value. Particular resonances have been established in the system.

Determine the bandwidth. From (11.59), for the bandwidth frequency limits we get:

$$Y_{21n}(A\omega) = \frac{2A}{\sqrt{\left(1 + A^2 - \xi_{\lim}^2\right)^2 + 4\xi_{\lim}^2}} = \frac{2A}{1 + A^2}\frac{1}{\sqrt{2}}.$$

Hence, we get the equation:

$$\xi_{\lim}^4 + 2\left(1 - A^2\right)\xi_{\lim}^2 - \left(1 + A^2\right)^2 = 0.$$



Its solution is the following:

$$\xi_{\lim 1,2} = \pm\sqrt{A^2 - 1 + \sqrt{2\left(A^4 + 1\right)}}.$$

Now, the frequency limits are:

$$\omega_{\lim 1,2} \approx \omega_0 \left[1 \mp \frac{1}{2Q}\sqrt{A^2 - 1 + \sqrt{2\left(A^4 + 1\right)}}\right].$$

The bandwidth is

$$B_0 = \frac{\omega_{\lim 2} - \omega_{\lim 1}}{\omega_0} = \frac{1}{Q}\sqrt{A^2 - 1 + \sqrt{2\left(A^4 + 1\right)}}\,. \tag{11.66}$$

For a very loose coupling ($K \le 0{,}01$), i.e. $Q_1 = Q_2 = Q$ we get

$$A = KQ << 0{,}01Q. \tag{11.67}$$

Substituting (11.67) into (11.66) we get

$$B_{0,707} = \frac{0{,}64\omega_0}{Q}.$$

This is less than with a single circuit $\left(B_0 = \frac{1}{Q}\right)$.

Having similarly calculated $B_{0,1} = \frac{3\omega_0}{Q}$, we get the following for the bandwidth ratio

$$K_B = \frac{B_{0,707}}{B_{0,17}} = \frac{0{,}64\omega_0 Q}{Q3\omega_0} = 0{,}213,$$

which is far greater than with a single circuit.

Thus, a higher selectivity is achieved even with particular resonances in a system of coupled circuits.

Fig. 11.30 shows the amplitude-frequency characteristic $Y_{21n}(\omega)$ for $A < 1$.

2. $A = 1$, the coupling is equal to optimal. From (11.61) we also get one solution $\xi_0 = 0$. That is, as in the case with $A < 1$ a resonance is achieved at one frequency $\omega_0 = \frac{1}{\sqrt{LC}}$. However, according to (11.65)

$$Y_{21n}(\omega_0)=\frac{2A}{1+A^2}=1,$$

i.e. the current in the secondary circuit reaches a maximum value. We have a total resonance in the system. The bandwidth is determined from (11.59) as:

$$Y_{21}(A,\omega)=\frac{2A}{\sqrt{\left(1+A^2-\xi_{\lim}^2\right)^2+4\xi_{\lim}^2}}=\frac{1}{\sqrt{2}}.$$

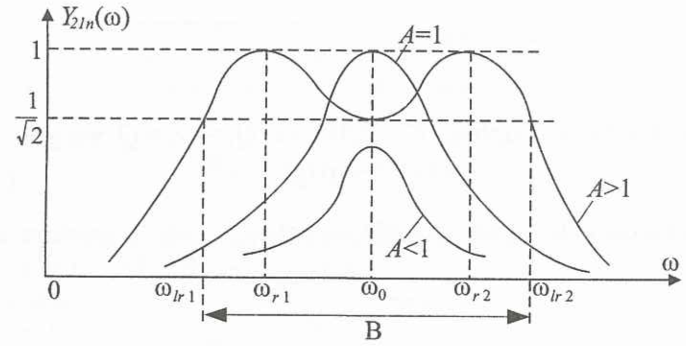



Fig. 11.30

From here we get the equation:

$$\left(\xi_{\lim}^2+1-A^2\right)^2=4A^2.$$

Its solution is:

$$\xi_{\lim 1,2}=\pm\sqrt{A^2+2A-1}.$$

The frequency limits are:

$$\omega_{\lim 1,2}=\omega_0\left[1\mp\frac{1}{2Q}\sqrt{A^2+2A-1}\right].$$

The bandwidth is

$$B_0=\frac{\omega_{\lim 2}-\omega_{\lim 1}}{\omega_0}=\frac{1}{Q}\sqrt{A^2+2A-1}.$$



For $A = 1$ $B_{0,707}=\frac{\sqrt{2}\omega_0}{Q}=\frac{1,41\omega_0}{Q}$. It is greater than with a single circuit $B_{0,707}=\frac{\omega_0}{Q}$. Having similarly calculated $B_{0,1}=\frac{4,44\omega_0}{Q}$ we get the following expression for the bandwidth ratio

$$K_B=\frac{B_{0,707}}{B_{0,17}}=\frac{1,41\omega_0 Q}{Q4,44\omega_0}=0,318.$$

This is greater than with a single circuit and with coupled circuits for $A<1$.

The amplitude-frequency characteristic $Y_{21n}(\omega)$ for $A = 1$ is given in Fig. 11.30.

3. $A > 1$. The coupling is tighter than optimal. There are three resonance frequencies on the basis of (11.63) and (11.64). For $\xi = 0$ at the frequency $\omega_0$, we get from (11.59):

$$Y_{21n}(\omega_0)=\frac{2A}{1+A^2}<1. \qquad (11.68)$$

That is if $A$ exceeds 1 at the frequency $\omega_0$, a dip occurs in the curve of the amplitude-frequency characteristic, and the current in the secondary circuit reduces.

Determine the value of $A$, with which the current reduces to $\frac{1}{\sqrt{2}}=0,707$ of the maximum value. From (11.68) we get

$$Y_{21}(\omega_0)=\frac{2A}{1+A^2}=\frac{1}{\sqrt{2}}.$$

Hence

$$A=\sqrt{2}+1=2,41.$$

That is for $A = 2,41$ the bandwidth, according to Fig. 11.30, is still continuous. For $A > 2,41$ it breaks down into two parts.

From (11.59) for $\xi_{1,2}=\pm\sqrt{A^2-1}$ we get

$$Y_{21n}(\omega_{1,2})=\frac{2A}{\sqrt{\left(1+A^2-A+1\right)^2+4\left(A^2-1\right)}}=1.$$

That is the current reaches a maximum value. Complex resonances have been established in the system. The bandwidth corresponding to $A = 2,41$ is determined as:

$$B_{0,707} = \frac{3,1\omega_0}{Q}.$$

It's much greater than at $A = 1$. Having similarly calculated $B_{0,707} = \frac{7,28\omega_0}{Q}$ we get the following expression for the bandwidth ratio

$$K_{\Pi} = \frac{B_{0,707}}{B_{0,17}} = \frac{3,1\omega_0 Q}{Q 7,28\omega_0} = 0,426,$$

which is greater than at $A = 1$.

A system of two coupled circuits is the simplest. Multiply coupled oscillatory circuits consisting of three or more circuits have wide application. Multiply coupled circuits allow higher selectivity. Table 11.1 represents dependence between the bandwidth ratio and the number of circuits in a system; we can see that with $n$ = 10 the frequency characteristic is close to the ideal one.

*Table 11.1*

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| $K_n$ | 0,1 | 0,426 | 0,64 | 0,81 | 0,89 | 0,95 | 0,98 | 0,99 | ~1,0 | ~1,0 |

**Example 5**

There is a system of two coupled circuits (Fig. 11.31) with identical parameters:

$$R_1 = R_2 = R = 24\ \Omega;\ C_1 = C_2 = C = 1,2\ \text{nF};\ C_{12} = 18\ \text{nF};$$

$$L_1 = L_2 = L = 0,5\ \text{mH}.$$

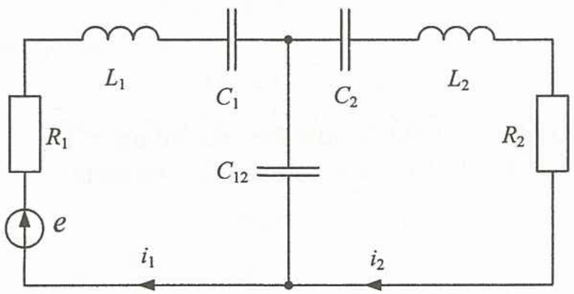



Fig. 11.31



Determine the resonance frequency, the quality factor $Q$, the coupling coefficient $k$, the coupled factor $A$ and the maximum current $I_{2mm}$ for $E = 1$ V.

***Solution***

As the circuits have the same parameters, the resonance frequencies and the quality factors of the circuits are also identical:

$$f_0 = \frac{1}{2\pi\sqrt{LC_{11}}};\ Q = \frac{1}{R}\sqrt{\frac{L}{C_{11}}};\ \rho_1 = \rho_2 = \rho.$$

Here $C_{11}, C_{22}$ — self capacity of the circuits

$$C_{11} = C_{22} = \frac{C_1 C_{12}}{C_1 + C_{12}} = \frac{1,2 \cdot 18}{1,2 + 18} = 1125\ \text{pF}.$$

Then

$$f_0 = \frac{1}{2\pi\sqrt{0,5\cdot10^{-3}\cdot1125\cdot10^{-12}}} = 21,23\ \text{kHz};$$

$$Q = \frac{1}{24}\sqrt{\frac{0,5\cdot10^{-3}}{1125\cdot10^{-12}}} = 27,8.$$

The coupling coefficient

$$k = \frac{x_c}{\sqrt{\rho_1\rho_2}} = \frac{x_c}{\rho};\quad x_c = \frac{1}{\omega_0 C_{12}}.$$

Then

$$k = \frac{1\cdot\omega_0 C_{11}}{\omega_0 C_{12}\cdot 1} = \frac{C_{11}}{C_{12}} = \frac{1125\cdot10^{-12}}{18\cdot10^{-9}} = 0,0625.$$

The coupling factor:

$$A = kQ = 0,0625\cdot27,8 = 1,74.$$

The maximum current in the secondary circuit:

$$I_{2mm} = \frac{E}{2R} = \frac{1}{2\cdot24} = 20,8\ \text{mA}.$$

**Example 6**

The antenna circuit I (Fig. 11.32) is inductively connected to the input circuit II of an amplifier. Both circuits are tuned to resonance to the frequency of the received signal $\omega_0 = 2,5\cdot10^6\ \text{s}^{-1}$.

The EMF $E_1 = 100$ mcV is induced in the antenna circuit. The parameters of the circuit are: $R_1 = 10\ \Omega$; $R_2 = 20\ \Omega$; $L_1 = 200$ mcH; $L_2 = 400$ mcH the coupling coefficient $k = 0{,}03$.

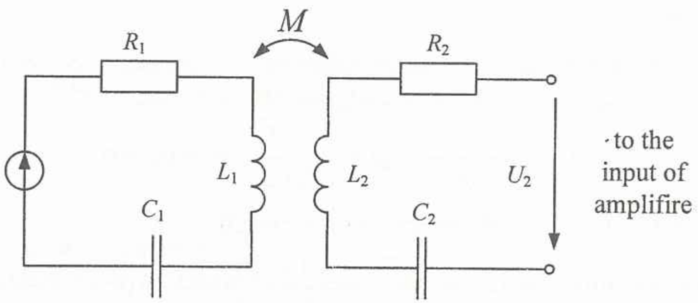



Fig. 11.32

Considering the input impedance of the amplifier infinite, determine: the capacitances $C_1$ and $C_2$, the qualities factors $Q_1$ and $Q_2$ the mutual inductance $M$; the current in the second circuit $I_2$; the voltage at the output of the amplifier $U_{20}$ at the frequency $f_0$, the coupled frequencies $\omega_1$, and $\omega_2$; the current $I_2$ and the voltage $U_2$ at the output of the amplifier at these frequencies.

***Solution***

The resonance frequency of the circuits:

$$\omega_{01} = \frac{1}{\sqrt{L_1 C_1}} = \omega_{02} = \frac{1}{\sqrt{L_2 C_2}} = \omega_0 = 2{,}5 \times 10^{-6}\ \text{s}^{-1}.$$

Hence

$$C_1 = \frac{1}{\omega_0^2 L_1} = \frac{1}{\left(2{,}5 \cdot 10^6\right)^2 \cdot 200 \cdot 10^{-6}} = 800\text{pF};$$

$$C_2 = \frac{1}{\omega_0^2 L_2} = \frac{1}{\left(2{,}5 \cdot 10^6\right)^2 \cdot 400 \cdot 10^{-6}} = 400\text{pF}.$$

The $Q$-factors of the circuits

$$Q_1 = \frac{1}{R_1}\sqrt{\frac{L_1}{C_1}} = \frac{1}{10}\sqrt{\frac{200 \cdot 10^{-6}}{800 \cdot 10^{-12}}} = 50;$$



$$Q_2 = \frac{1}{R_2}\sqrt{\frac{L_2}{C_2}} = \frac{1}{20}\sqrt{\frac{400 \cdot 10^{-6}}{400 \cdot 10^{-12}}} = 50.$$

I.e. the $Q$-factors of the circuits are the same.

The mutual inductance $M$ will be determined from the expression for the coupling coefficient:

$$k = \frac{M}{\sqrt{L_1 L_2}}.$$

Then

$$M = k\sqrt{L_1 L_2} = 0{,}03\sqrt{200 \cdot 10^{-6} \cdot 400 \cdot 10^{-6}} = 8{,}5\ \mu\text{H}.$$

The generalized detuning of the circuit:

$$\xi_1 = Q_1\left(\frac{\omega_{01}}{\omega_0} - \frac{\omega_0}{\omega_{01}}\right);\ \xi_2 = Q_2\left(\frac{\omega_{02}}{\omega_0} - \frac{\omega_0}{\omega_{02}}\right).$$

For $\omega_{01} = \omega_{02} = \omega_0$ the generalized circuit detuning $\xi_1 = \xi_2 = \xi = 0$. The coupling factor:

$$A = \frac{x_c}{\sqrt{R_1 R_2}} = \frac{\omega_0 M}{\sqrt{R_1 R_2}} = \frac{2{,}5 \cdot 10^6 \cdot 8{,}8 \cdot 10^{-6}}{\sqrt{10 \cdot 20}} = 1{,}5.$$

The current flowing in the secondary circuit:

$$I_{m2} = \frac{E_1 A}{\sqrt{R_1 R_2}\sqrt{\left(1 + A^2 - \xi^2\right)^2 + 4\xi^2}}.$$

For $\xi = 0$:

$$I_2 = \frac{E_1 A}{\left(1 + A^2\right)\sqrt{R_1 R_2}} = \frac{100 \cdot 10^{-6} \cdot 1{,}5}{\left(1 + 1{,}5^2\right)\sqrt{10 \cdot 20}} = 3{,}26\ \mu\text{A}.$$

The voltage at the output of the amplifier:

$$U_2 = I_2 \omega_0 L_2 = 3{,}26 \cdot 10^{-6} \cdot 2{,}5 \cdot 10^6 \cdot 400 \cdot 10^{-6} = 3{,}27\ \text{mV}.$$

The coupling factor $A$ = 1,5, i.e. it is optimal. The system has three resonance frequencies. Two of them, at which the current $I_{2mm}$ is maximum, are called the coupling frequencies.

Determine the attenuation of the circuits:

$$d = \frac{1}{Q} = \frac{1}{50} = 0,02 = d_1 = d_2.$$

The critical coupling coefficient for a system of coupled circuits with three resonance frequencies is determined by the expression:

$$k_{cr} = \sqrt{\frac{d_1^2 + d_2^2}{2}} = \sqrt{\frac{0,02^2 + 0,02^2}{2}} = 0,02.$$

The coupling coefficient is specified as $k = 0,03 > k_{cr}$, i.e. there are the coupled frequencies $\omega_{01}$, $\omega_{02}$ that are determined as:

$$\omega_{01} = \frac{\omega_0}{\sqrt{1+\sqrt{k^2 - k_{cr}^2}}} = \frac{2,5\cdot 10^6}{\sqrt{1+\sqrt{0,03^2 - 0,02^2}}} = 2,47\cdot 10\ \text{s}^{-1};$$

$$\omega_{02} = \frac{\omega_0}{\sqrt{1-\sqrt{k^2 - k_{cr}^2}}} = \frac{2,5\cdot 10^6}{\sqrt{1-\sqrt{0,03^2 - 0,02^2}}} = 2,53\cdot 10\ \text{s}^{-1}.$$

From the formula

$$\omega_{02} = \omega_0\left(1 + \frac{1}{2Q}\xi\right)$$

we get the generalized circuit detuning

$$\xi = \frac{2Q(\omega_{02} - \omega_0)}{\omega_{02}} = \frac{2\cdot 50\cdot\left(2,53\cdot 10^6 - 2,5\cdot 10^6\right)}{2,53\cdot 10^6} = 1,2.$$

The current $I_{m2}$ in the secondary circuit for $\xi = 1,2$ is

$$I_{m2} = \frac{E_1 A}{\sqrt{R_1 R_2}\sqrt{\left(1 + A^2 - \xi^2\right)^2 + 4\xi^2}} =$$

$$= \frac{100\cdot 10^{-6}\cdot 1,5}{\sqrt{10\cdot 20}\sqrt{\left(1+1,5^2 - 1,2^2\right)^2 + 4\cdot 1,2^2}} = 3,35\cdot 10^{-6}.$$